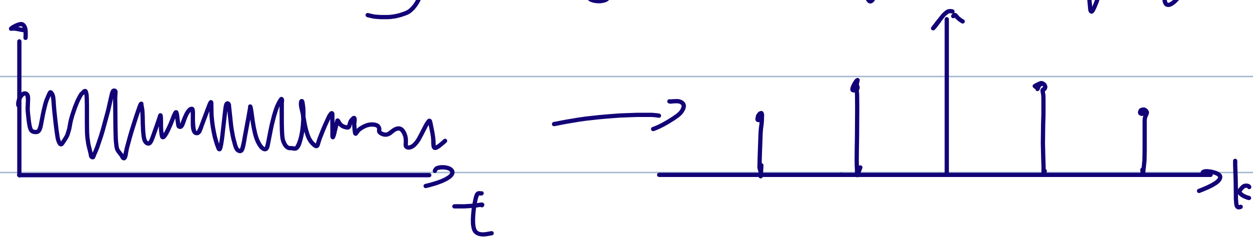
Previously, we showed that any periodic function can be represented as a series of harmonics:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where: $a_k = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt$

↑ Representation of a time dependent signal w/ harmonic sinusoids where, the harmonic sinusoids are essentially integers at different frequencies.



But then, for the representation of the signal on "true freq" ω, & $\because \omega = \frac{2\pi}{T} \cdot k = \omega_0 \cdot k$, we must "normalize" amplitudes by T so that the total "energy" of the signal is consistent.

Also Previously, the "harmonic condition" is matched for Amplitudes only when we use this assumption:

$$@ k=n, \int_0^T e^{j(k-n)\omega_0 t} dt = T, \quad @ k=\text{else}, 0$$

When there is not a defined "T" in the original function, the $\int_0^T e^{jk\omega_0 t} dt$ is not discrete at $k=n$: aka k is continuous.

$\therefore$

rewrite: $a_k = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t}$

define: $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

compare w/ ak: $a_k = \frac{1}{T} X(jk\omega_0)$

$\therefore x(t) = \sum_{-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}$

$(T \to \infty) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$

$\hookrightarrow X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$

$\Uparrow$ Anyways, above is extra stuff for forier transform, not in the scope of the course

Previously, in our string example:

$y_n(x,t) = A_n \sin(\frac{n\pi}{L}x)\cos(\omega_n t)$

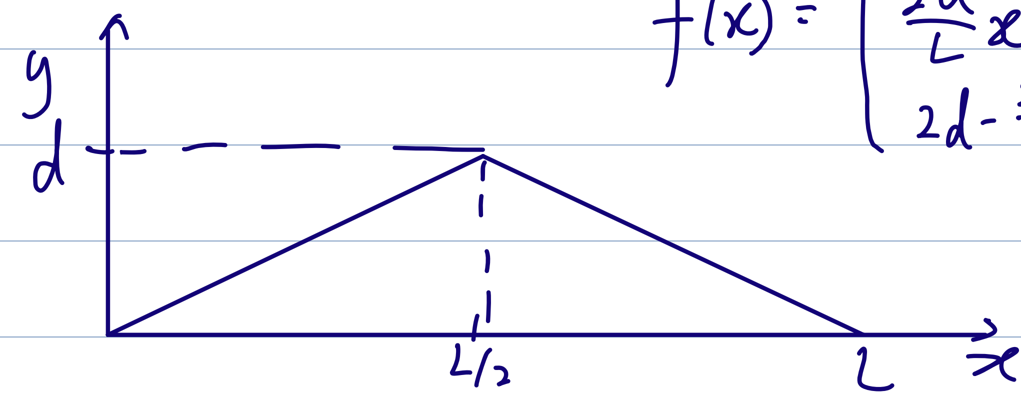
$y(x,0) = \sum_n A_n \sin(\frac{n\pi}{L}x) = f(x)$ [superposition of normal modes]

Any string w/ fixed end points $[f(0)=f(L)=0]$ is:

$$\begin{cases} f(x) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right) \\ A_n = \frac{2}{L}\int_0^L dx \sin\left(\frac{n\pi}{L}x\right) f(x), \quad n \in \{+\mathbb{Z}\} \end{cases}$$

example:

$t=0$:



$$f(x) = \begin{cases} \frac{2d}{L}x, & x \in [0, L/2] \\ 2d - \frac{2d}{L}x, & x \in [L/2, L] \end{cases}$$

$$A_n = \frac{2}{L}\left[\int_0^{L/2} dx \frac{2dx}{L}\sin\left(\frac{n\pi}{L}x\right) + \int_{L/2}^{L} dx \left(2d - \frac{2d}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\right]$$

$$= \frac{2}{L}\left[\frac{2d}{L}\int_0^{L/2} x\sin\left(\frac{n\pi}{L}x\right)dx + 2d\int_{L/2}^{L}\sin\left(\frac{n\pi}{L}x\right)dx\right.$$

$$\left. - \frac{2d}{L}\int_{L/2}^{L} x\sin\left(\frac{n\pi}{L}x\right)dx\right]$$

$$\int x\sin(ax)\,dx = \int u\,dv = uv - \int v\,du$$

$u = x$, $du = dx$

$dv = \sin(ax)\,dx$, $v = \frac{-\cos(ax)}{a}$

$$= -\frac{x}{a}\cos(ax) + \int \frac{\cos(ax)}{a}dx$$

$$= -\frac{x}{a}\cos(ax) + \frac{1}{a^2}\sin(ax)$$

$$\text{So: } \int_0^{L/2} x\sin\left(\frac{n\pi}{L}x\right)dx = -\frac{L}{n\pi}x\cos\left(\frac{n\pi}{L}x\right) + \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}x\right)\Bigg|_0^{L/2}$$

$$= -\frac{L}{n\pi}\frac{L}{2}\cos\left(\frac{n\pi}{2}\right) + \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

$$+ \cancel{\frac{L}{n\pi}0\cos(0)} \to 0 \; - \cancel{\frac{L^2}{(n\pi)^2}\sin 0} \to 0$$

$$= \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) - \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$\int_{L/2}^{L} x\sin\left(\frac{n\pi}{L}x\right)dx = -\frac{L}{n\pi}x\cos\left(\frac{n\pi}{L}x\right) + \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}x\right)\Bigg|_{L/2}^{L}$$

$$= -\frac{L}{n\pi}L\cos(n\pi) + \cancel{\frac{L^2}{(n\pi)^2}\sin(n\pi)} \to 0$$

$$+\frac{L}{n\pi}\frac{L}{2}\cos\left(\frac{n\pi}{2}\right) - \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\int_{L/2}^{L} \sin\left(\frac{n\pi}{L}x\right)dx = -\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\Bigg|_{L/2}^{L}$$

$$= -\frac{L}{n\pi}\cos(n\pi) + \frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

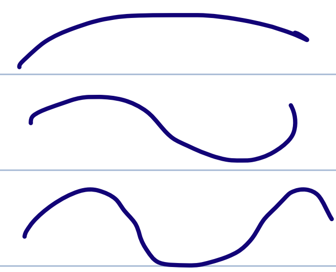
$$\text{plug in: } A_n = \frac{2}{L}\Bigg[\frac{2d}{\cancel{L}}\cdot\frac{\cancel{L^2}}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) \qquad \cancel{+2d\cdot\left(-\frac{L}{n\pi}\cos(n\pi)\right)}$$

$$-\frac{2d}{L}\left(\cancel{-\frac{L^2}{n\pi}\cos(n\pi)} + \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right)\Bigg]$$

$$= \frac{2}{L}\left[\frac{2dL}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) + \frac{2dL}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

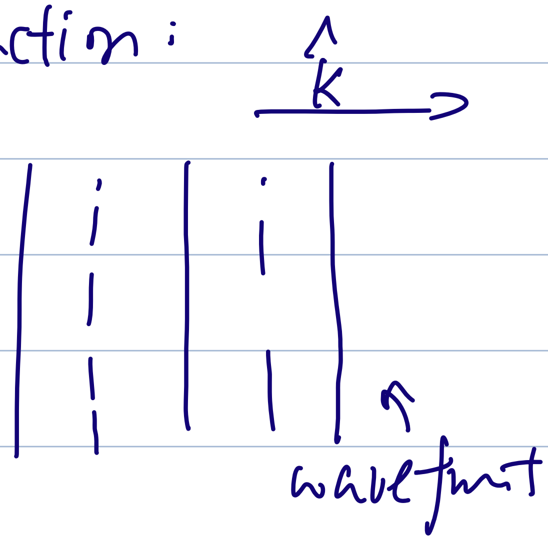
$$A_n = \frac{8d}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

Evaluation:

$n=$	A_n	$\sin\left(\frac{n\pi}{2}x\right)$
1	$A_n = \frac{8d}{\pi^2}$	
2	0	
3	$-8d/(3\pi)^2$	
4	0	$\vdots$
5	$8d/(5\pi)^2$	

Interference & diffraction:

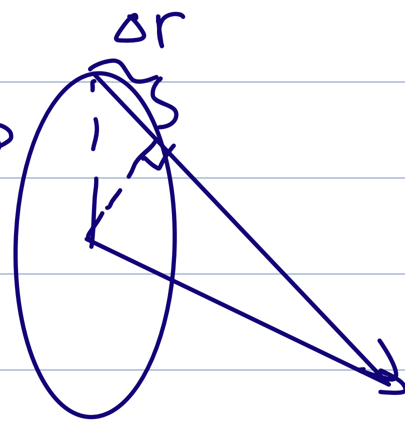
plane waves

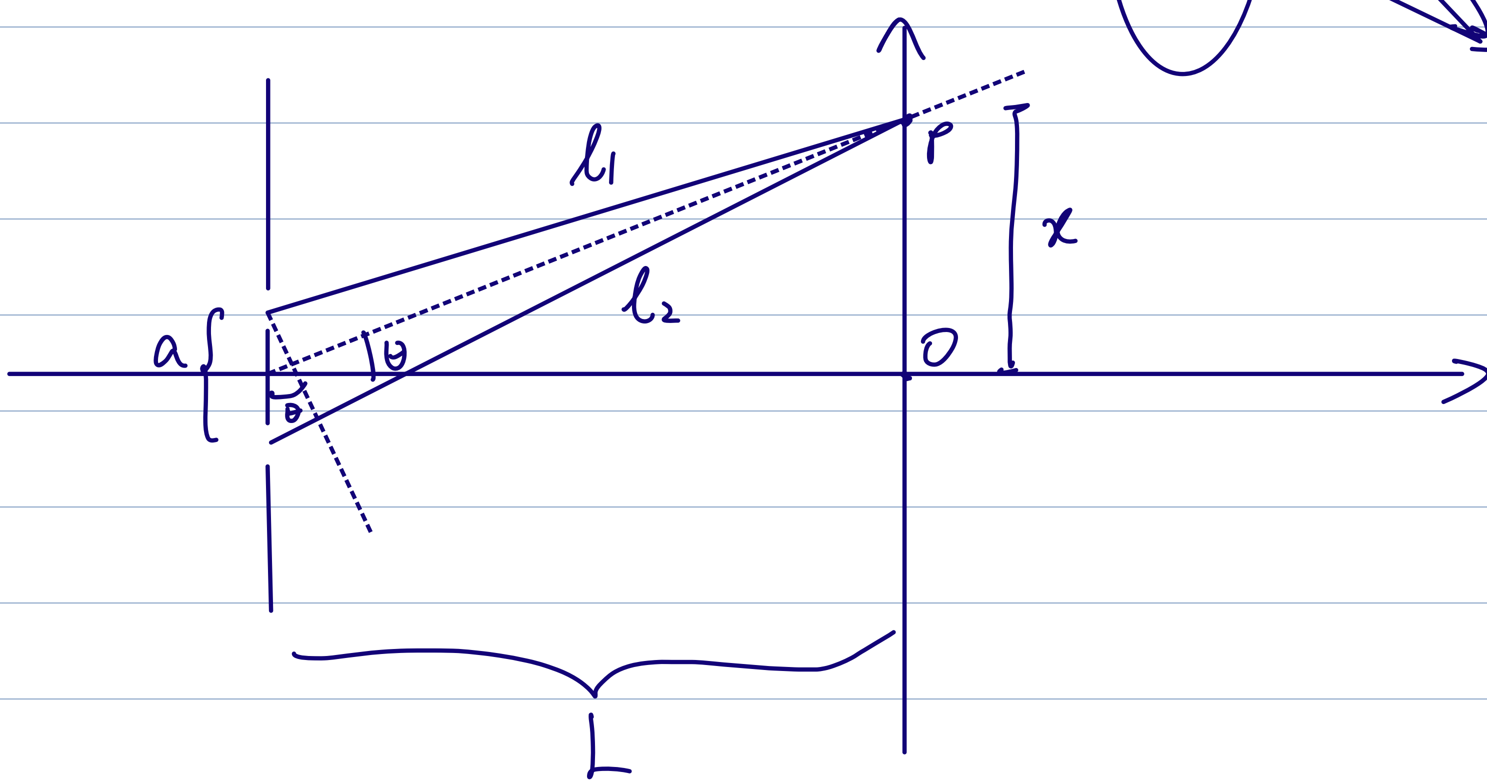


huygens:

any point on the wave front can be a source of a new wave

Diffraction is interference at a small but dense scale →





$$R = A\left[\cos(\omega t - k l_1) + \cos(\omega t - k l_2)\right]$$

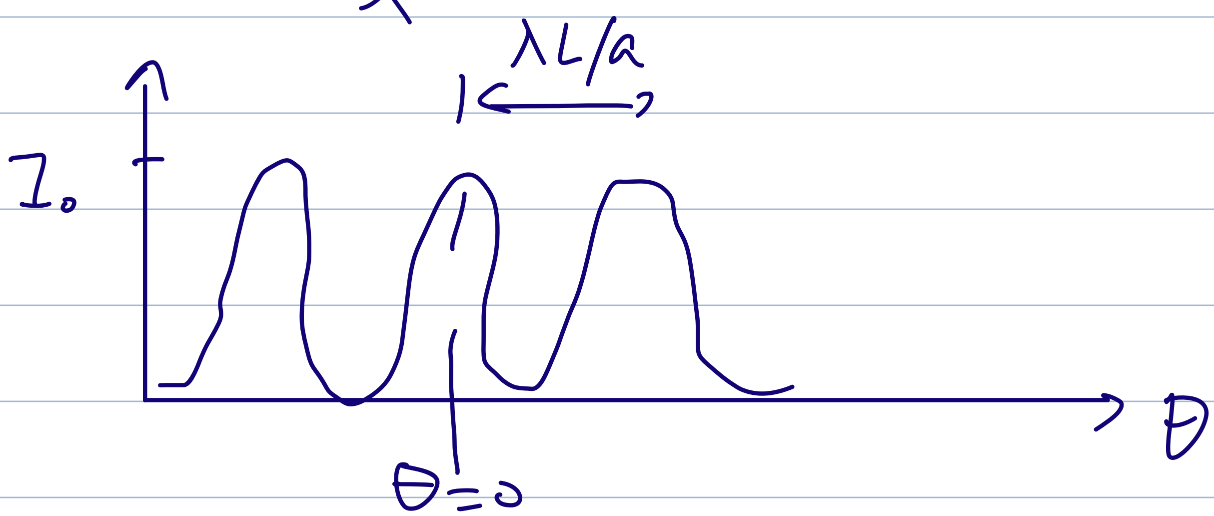
$$= 2A\cos\left(\omega t - k(l_1+l_2)/2\right)\cdot\cos\left(k(l_2-l_1)/2\right)$$

$$I \propto \tfrac{1}{2}R^2 = 2A^2\cos^2\left[k(l_2-l_1)/2\right]$$

$$l_2 - l_1 \triangleq \Delta l = a\sin\theta = a\theta$$

$$\therefore\ I(\theta) = I_0\cos^2\left(\frac{ka\theta}{2}\right) = I_0\cos^2\left(\frac{\pi a\theta}{\lambda}\right)$$

$$k = \frac{2\pi}{\lambda}$$



$$x = L\cdot\tan\theta = L\theta$$